4.6 – Dimension

Due Sun

Theorem 4.6.2 Let V be a finite-dimensional vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis for V.

1. If a set in V has more than n vectors, then it is linearly dependent.
2. If a set in V has fewer than n vectors, then it does not span V.

The proof of this essentially involves counting variables and equations in a linear system.

Theorem 4.6.1 All bases for a finite-dimensional vector space have the same number of vectors.

Definition: The **dimension** of a finite-dimensional vector space V is denoted by $\dim(V)$ and is defined to be the number of vectors in a basis for V (in some physical contexts, "dimension" is referred to as **degrees of freedom**). In addition, the zero vector space is defined to have dimension zero.

(Recall that the basis for $\{\vec{0}\}$ is $\emptyset$.

Find a basis for the solution space of each homogeneous linear system, and find the dimension of that space.

#1

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ -2x_1 - x_2 + 2x_3 &= 0 \\ -x_1 + x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1: x_1 = x_3$, $R_2: x_2 = 0$

Solution set: $\left\{\vec{x} \mid \vec{x} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t, \ t \in \mathbb{R} \right\}$

Basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$, Dim: 1

#5

$$2x_1 - 6x_2 + 2x_3 = 0$$
$$3x_1 - 9x_2 + 3x_3 = 0$$

$\rightarrow \begin{bmatrix} 2 & -6 & 2 \\ 3 & -9 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 = 3x_2 - x_3$ so $\vec{x} = \begin{bmatrix} 3s - t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} t$

Basis: $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$, dim: 2

#8 In each part, find a basis for the given subspace of R^4, and state its dimension.

a. All vectors of the form $(a,\ b,\ c,\ 0)$.

b. All vectors of the form $(a,\ b,\ c,\ d)$, where $d = a + b$ and $c = a - b$.

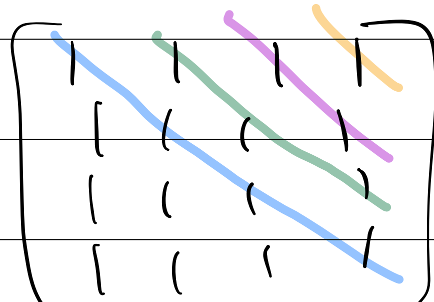
c. All vectors of the form $(a,\ b,\ c,\ d)$, where $a = b = c = d$.

a) $\{(1,0,0,0), (0,1,0,0), (0,0,1,0)\}$, dim: 3

b) $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \\ a-b \\ a+b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} a + \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} b$

Basis: $\{(1,0,1,1), (0,1,-1,1)\}$, dim: 2

c) Basis: $\{(1,1,1,1)\}$ dim: 1



#9 Find the dimension of each of the following vector spaces.
a. The vector space of all diagonal $n \times n$ matrices.
b. The vector space of all symmetric $n \times n$ matrices.
c. The vector space of all upper triangular $n \times n$ matrices.

$$\begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots & d_n \end{bmatrix}$$

a) $n \times n$ diagonal matrices have up to n nonzero entries, so the dimension is n.

(b) & (c): We only need to consider entries on and above the main diagonal, giving degrees of freedom.

$$n + (n-1) + (n-2) + \cdots + 3 + 2 + 1$$
$$= \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \leftarrow \text{dim}$$

Theorem 4.6.4 Let V be an n-dimensional vector space, and let S be a set in V with exactly n vectors. Then S is a basis for V if and only if S spans V or S is linearly independent.

3 Conditions:

- correct # vectors (dim)
- spanning
- lin. indep.

If any 2 of these are true, then the third is guaranteed.

Theorem 4.6.5 Let S be a finite set of vectors in a finite-dimensional vector space V.

a) If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.

b) If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

$\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$

#13 Find standard basis vectors for R^4 that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for R^4.

$\mathbf{v}_1 = (1,\ -4,\ 2,\ -3)$, $\mathbf{v}_2 = (-3,\ 8,\ -4,\ 6)$

$$\begin{bmatrix} 1 & -3 & 1 & 0 & 0 & 0 \\ -4 & 8 & 0 & 1 & 0 & 0 \\ 2 & -4 & 0 & 0 & 1 & 0 \\ -3 & 6 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & -1/3 \\ 0 & 0 & 0 & 1 & 0 & -4/3 \\ 0 & 0 & 0 & 0 & 1 & 2/3 \end{bmatrix}$$

Columns: $\vec{v}_1$, $\vec{v}_2$, $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, $\vec{e}_4$ — using pivots

The desired basis is $\{\vec{v}_1, \vec{v}_2, \vec{e}_2, \vec{e}_3\}$.

#17 Find a basis for the subspace of R^3 that is spanned by the vectors $\mathbf{v}_1 = (1,\ 0,\ 0)$, $\mathbf{v}_2 = (1,\ 0,\ 1)$, $\mathbf{v}_3 = (2,\ 0,\ 1)$, $\mathbf{v}_4 = (0,\ 0,\ -1)$.

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

use pivot columns

$\{\vec{v}_1, \vec{v}_2\}$.

#20 In each part, let T_A be multiplication by A and find the dimension of the subspace of R^4 consisting of all vectors x for which $T_A(\mathbf{x}) = \mathbf{0}$.

— kernel

a. $\begin{bmatrix} 1 & 0 & 2 & -1 \\ -1 & 4 & 0 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

That is, find the dimension of the kernel.

The process is the same as above.